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# A conditional stability criterion based on generalized energies

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An energy criterion for conditional stability is proposed, based on the definition of generalized energies, obtained through a perturbation of the classical $L^2$ (kinetic) energy. This perturbation is such that the contribution of the linear term in the perturbation equation to the generalized energy time derivative is negative definite. A critical amplitude threshold is then obtained by imposing the monotonic decay of the generalized energy. The capabilities of the procedure are appraised through the application to three different low-dimensional models. The effects of different choices in the construction of the generalized energy on the prediction of the critical amplitude threshold in the subcritical regime are also discussed.

## 1. Introduction

Many shear flows are characterized by the so-called subcritical instability, i.e. instability occurring at Reynolds numbers, $Re$, for which the laminar base flow is linearly stable. In this regime, the stability of the base flow is guaranteed only if the initial amplitude of a given disturbance is lower than a critical threshold value. This value is a decreasing function of $Re$ between $Re_g$ and $Re_L$, $Re_L$ being the critical Reynolds number predicted by linear stability and $Re_g$ being the Reynolds number of global stability, i.e. for $Re < Re_g$ all disturbances, whatever their initial amplitude, decay asymptotically in time. For most of the flows characterized by subcritical instability, the classical energy stability theory predicts a critical Reynolds number ($Re_E$) which is significantly lower than $Re_g$. This discrepancy is due, first, to this theory requiring the disturbance energy to decrease monotonically to zero; however, the non-normality of the linearized Navier–Stokes (NS) operator leads to the well-known transient growth of the disturbance energy, which eventually may decrease to zero. Furthermore, for many flows the nonlinear part of the NS operator and, consequently, the amplitude of the disturbance, do not play a role in the computation of $Re_E$.

The aim of the present work is to investigate whether the predictions of the energy theory may be improved if the problems related to the non-normality of the linear operator are by-passed and the amplitude of the disturbance is taken into account. To this end, generalized energies can be defined, such that the contribution to their time derivative of the linear term in the perturbation equation is negative definite. Generalized energies have already been introduced in hydrodynamic stability problems, mainly to extend the energy stability theory to flows involving particular phenomena, such as heat transfer or convection in porous media (see e.g. Straughan 2004 and Joseph 1976). Furthermore, methods based on Lyapunov functions (generalized energies) have been used to find analytical bounds for conditional stability (see e.g. Kaiser & Mulone 2005). An *exotic* energy, characterized by a negative-definite linear contribution to its time derivative, has been introduced in Dauchot & Manneville (1997) for a particular two-dimensional dynamic system, in an investigation of the importance of nonlinear effects in the subcritical transition regime. In the present study, a procedure is proposed to construct generalized energies through a perturbation of the $L^2$ metric. If these energies are used in the framework of the energy stability theory, in place of the kinetic energy, a criterion for conditional stability is obtained, which can be used to numerically compute critical threshold values for the disturbance amplitude in the subcritical regime. The proposed procedure is then applied to different low-dimensional models: the two-dimensional model of Dauchot & Manneville (1997), the four-dimensional system describing a self-sustained process in wall turbulence (Waleffe 1995), and the four-dimensional model for a transitional Couette-like shear flow (Waleffe 1997). The results are compared to those from numerical simulations reported in Cossu (2005, 2004).

## 2. Energy stability criterion with a generalized energy

We analyse how a generalized energy stability theory can be developed by replacing the kinetic energy with $\mathscr{E}$, which is a positive-definite bilinear form given by

$$\mathscr{E}(\boldsymbol{u}) = \tfrac{1}{2}\langle \boldsymbol{u}, \boldsymbol{u}\rangle_{\mathscr{E}} \tag{2.1}$$

in which $\boldsymbol{u}$ is a given disturbance of the base flow and $\langle\cdot,\cdot\rangle_{\mathscr{E}}$ denotes the scalar product in a new metric, which will be defined in § 3.

In analogy with the derivation of the Reynolds–Orr equation, the functional expressing the time derivative of the new energy is obtained from the equation governing the evolution of the disturbance. In the present case, as the nonlinear term contribution is typically non-zero, this functional is not homogeneous: a quadratic part is present, which is related to the linear operator in the disturbance equation, $\boldsymbol{L}$, and is expressed as: $\mathscr{L}(\boldsymbol{u}) = \langle \boldsymbol{u}, \boldsymbol{L}\boldsymbol{u}\rangle_{\mathscr{E}}$. The cubic part, related to the nonlinear term in the disturbance equation, $\boldsymbol{N}(\boldsymbol{u})$, is $\mathscr{N}(\boldsymbol{u}) = -\langle \boldsymbol{u}, \boldsymbol{N}(\boldsymbol{u})\rangle_{\mathscr{E}}$. Hence, we finally have

$$\frac{\mathrm{d}\mathscr{E}(\boldsymbol{u})}{\mathrm{d}t} = \mathscr{L}(\boldsymbol{u}) + \mathscr{N}(\boldsymbol{u})$$

and the stability criterion requires the time derivative of $\mathscr{E}$ to be negative, i.e. the generalized energy to decay monotonically. The two parts of the functional scale differently with respect to the disturbance amplitude: for a generic disturbance $\boldsymbol{u}$

$$\frac{\mathrm{d}\mathscr{E}(\alpha\boldsymbol{u})}{\mathrm{d}t} = \alpha^2\mathscr{L}(\boldsymbol{u}) + \alpha^3\mathscr{N}(\boldsymbol{u}) \qquad \forall\ \alpha \in \mathbb{R}. \tag{2.2}$$

Thus, for a very small disturbance the only significant term is the quadratic one, while the cubic one progressively becomes dominant as the disturbance amplitude increases. This implies, first, that a stability criterion based on the monotonic decay of the new energy is meaningful only if the quadratic part $\mathscr{L}$ is negative definite. Thus, the new energy must be constructed in such a way that $\mathscr{L}$ be negative definite. This is not the case for the classical energy, if the Reynolds number is larger than $Re_E$. As discussed in the Appendix, independently of the metric, $\mathscr{L}$ cannot be negative definite if $Re > Re_L$, and thus the proposed criterion is clearly restricted to $Re < Re_L$.

The second consequence of the inhomogeneity of the two contributions to the time derivative of $\mathscr{E}$ is that the stability criterion based on the monotonic decay of the new energy will be conditioned by the amplitude of the initial disturbance. Let us define the functional

$$\mathscr{F}(\boldsymbol{u}) = -\frac{\mathscr{N}(\boldsymbol{u})}{\mathscr{L}(\boldsymbol{u})\sqrt{\mathscr{E}(\boldsymbol{u})}},$$

which is homogeneous of degree 0. Let us assume $\mathscr{F}(\boldsymbol{u})$ to be bounded, and define

$$\frac{1}{\alpha} = \sup_{\boldsymbol{u}} \mathscr{F}(\boldsymbol{u}). \tag{2.3}$$

It follows that, for the range of Reynolds numbers for which the contribution of the linear terms is negative definite, the following inequality holds:

$$\mathscr{N}(\boldsymbol{u}) \leqslant -\frac{1}{\alpha}\mathscr{L}(\boldsymbol{u})\sqrt{\mathscr{E}(\boldsymbol{u})},$$

and thus:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{E}(\boldsymbol{u}) \leqslant \mathscr{L}(\boldsymbol{u})\left(1 - \frac{1}{\alpha}\sqrt{\mathscr{E}(\boldsymbol{u})}\right). \tag{2.4}$$

This gives a conditional monotonic stability criterion: if the initial generalized energy is lower than $\alpha^2$, it monotonically decays to zero.

In general, the threshold value $\alpha$ will be a function of the Reynolds number, and the classical energy stability theory may be considered as a special case giving a threshold value that is infinite for Reynolds numbers below $Re_E$ and zero otherwise. By using the $L^2$ energy, the contribution of the nonlinear term, $\mathscr{N}(\boldsymbol{u})$, vanishes and, thus, the time evolution of energy does indeed becomes independent of $\alpha$ (see (2.2)). Stability is hence obtained for all disturbance intensity only if $\mathscr{L}(\boldsymbol{u})$ is negatively defined, i.e. if $Re < Re_E$. We consider the more general case of a new energy functional which depends on the Reynolds number. In this way we can express a sufficiently general criterion of stability to retrieve the classical energy stability results for $Re = Re_E$ and to give a zero amplitude threshold for $Re$ approaching the linear critical value, $Re_L$.

## 3. Generalized energy construction

In this section we show how a generalized energy can be defined to meet the above requirements. To summarize, we search for a *positive-definite* bilinear form $\mathscr{E}$, having an associated *negative-definite* functional $\mathscr{L}$. It can be shown (see the Appendix) that the required condition for $\mathscr{L}$ to be negative definite automatically implies that $\mathscr{E}$ is positive definite, and, thus, in constructing the new energy, we need only consider satisfaction of the requirement on $\mathscr{L}$.

As stated in § 2, $\mathscr{E}$ is a generalized energy in a new metric, which is obtained through a perturbation of the $L^2$ metric as follows:

$$\langle \boldsymbol{u}, \boldsymbol{v}\rangle_{\mathscr{E}} = \langle \boldsymbol{u}, \boldsymbol{v}\rangle_{L^2} + \langle \boldsymbol{u}, \boldsymbol{v}\rangle_{\mathscr{P}}, \tag{3.1}$$

where $\langle\cdot,\cdot\rangle_{L^2}$ denotes the $L^2$ scalar product and $\langle\cdot,\cdot\rangle_{\mathscr{P}}$ the perturbation term.

The contribution of the linear term to the time variation of the generalized energy may then be rewritten as follows:

$$\mathscr{L}(\boldsymbol{u}) = \langle \boldsymbol{u}, \boldsymbol{L}\boldsymbol{u}\rangle_{\mathscr{E}} = \langle \boldsymbol{u}, \boldsymbol{L}\boldsymbol{u}\rangle_{L^2} + \langle \boldsymbol{u}, \boldsymbol{L}\boldsymbol{u}\rangle_{\mathscr{P}} \tag{3.2}$$

where, under the same assumptions on boundary conditions used in the derivation of the Reynolds–Orr equation, $\langle \boldsymbol{u}, \boldsymbol{L}\boldsymbol{u}\rangle_{L^2}$ is the time variation of the classical energy. We search for a decomposition of the disturbance space, $\mathscr{D} = \mathscr{D}^+ \oplus \mathscr{D}^-$, such that the restriction of $\langle \boldsymbol{u}, \boldsymbol{L}\boldsymbol{u}\rangle_{L^2}$ to $\mathscr{D}^-$ is negative definite. Note that $\mathscr{D}^-$ does not necessarily contain all the disturbances satisfying this requirement. Also $\mathscr{D}^-$ must be invariant with respect to $\boldsymbol{L}$; this is not strictly needed in finite dimensions, as in all the cases considered here, but it is maintained in view of a formal extension of the method to infinite-dimensional space (Navier–Stokes equations). The linear operator $\boldsymbol{L}$ may thus be split as the sum of $\boldsymbol{L}^+ = \boldsymbol{L}(\pi_{\mathscr{D}^+}(\cdot))$ and $\boldsymbol{L}^- = \boldsymbol{L}(\pi_{\mathscr{D}^-}(\cdot))$, $\pi_{\mathscr{D}}^+$ and $\pi_{\mathscr{D}}^-$ being the projections on $\mathscr{D}^+$ and $\mathscr{D}^-$. Thus, $\boldsymbol{L}^+$ contains the part of $\boldsymbol{L}$ responsible for the linear transient growth of the classical energy. Hence, $\langle \boldsymbol{u}, \boldsymbol{L}\boldsymbol{u}\rangle_{L^2}$ can also be split in two parts: $\mathscr{V}^-(\boldsymbol{u}) = \langle \boldsymbol{u}, \boldsymbol{L}^-\boldsymbol{u}\rangle_{L^2}$ and $\mathscr{V}^+(\boldsymbol{u}) = \langle \boldsymbol{u}, \boldsymbol{L}^+\boldsymbol{u}\rangle_{L^2}$; $\mathscr{V}^-(\boldsymbol{u})$ is negative semi-definite, while $\mathscr{V}^+(\boldsymbol{u})$ is non-negative definite for $Re > Re_E$. Note that, as will be shown also in §4, this decomposition is not unique.

As previously stated, the perturbation term, $\langle \boldsymbol{u}, \boldsymbol{L}\boldsymbol{u}\rangle_{\mathscr{P}}$, should be chosen in order to render $\mathscr{L}(\boldsymbol{u})$ negative definite (for $Re < Re_L$). This can be done by choosing

$$\langle \boldsymbol{u}, \boldsymbol{L}\boldsymbol{u}\rangle_{\mathscr{P}} = -(\mathscr{V}^+(\boldsymbol{u}) + \varepsilon E(\pi_{\mathscr{D}^+}(\boldsymbol{u}))) \tag{3.3}$$

in which $\varepsilon$ is a positive free parameter. In practice, (see §4) the perturbation operator can be computed by numerically solving (3.3). From (3.2) and (3.3), it follows that $\mathscr{L}(\boldsymbol{u})$ is negative definite and the following inequality holds: $\mathscr{L}(\boldsymbol{u}) \leqslant -\varepsilon E(\pi_{\mathscr{D}^+}(\boldsymbol{u}))$.

## 4. Application to low-dimensional models

A low-dimensional model written in the following general form is considered:

$$\frac{\mathrm{d}\boldsymbol{U}}{\mathrm{d}t} = \boldsymbol{L}\,\boldsymbol{U} + N(\boldsymbol{U}). \tag{4.1}$$

in which $\boldsymbol{U}$ is the vector of the unknown variables. Attention is focused on the two-dimensional model proposed in Dauchot & Manneville (1997) (called DM hereafter), on the four-dimensional system describing a self-sustained process in wall turbulence (Waleffe 1995, W95), and on the four-dimensional model for a transitional Couette-like shear flow (Waleffe 1997, W97).

For the DM model the two relevant variables are denoted $\boldsymbol{u} = (u, v)^T$, and the linear operator in (4.1) is given by

$$\boldsymbol{L} = \begin{pmatrix} s_1 & 1 \\ 0 & s_2 \end{pmatrix}$$

where $s_1$ and $s_2$ are functions of a control parameter (the Reynolds number, $Re$); for linear stability both $s_1$ and $s_2$ must be negative. The nonlinear term is expressed as $N(\boldsymbol{u}) = (uv, -u^2)^T$. The dynamics of the model in the phase space has been investigated in Dauchot & Manneville (1997). The trivial solution $u = v = 0$ represents the laminar basic flow.

By using (2.1) and (3.1), the generalized energy can be written in a matrix form:

$$\mathscr{E} = \tfrac{1}{2}(\boldsymbol{U}^T(\boldsymbol{I} + \boldsymbol{P})\,\boldsymbol{U}) \tag{4.2}$$

in which $\boldsymbol{I}$ is the identity matrix, and $\boldsymbol{P}$ is the perturbation matrix. Thus, the generalized energy is defined once $\boldsymbol{P}$ is computed. By recalling the definition of $\mathscr{V}^+(\boldsymbol{u})$, (3.3) may also be rewritten in a matrix form as follows:

$$\boldsymbol{U}^T\,\boldsymbol{P}\,\boldsymbol{L}\boldsymbol{U} = -\boldsymbol{U}^T\boldsymbol{L}^+\boldsymbol{U} - \tfrac{1}{2}\varepsilon\boldsymbol{U}^T\,\boldsymbol{\Pi}^T\,\boldsymbol{\Pi}\boldsymbol{U} \tag{4.3}$$

where $\boldsymbol{\Pi}$ is the matrix projecting $\boldsymbol{U}$ on the subspace $\mathscr{D}^+$, i.e. $\boldsymbol{U}^+ = \boldsymbol{\Pi}\,\boldsymbol{U}$. In all the cases considered hereafter, $\boldsymbol{\Pi} = \boldsymbol{I}^+$, $\boldsymbol{I}^+$ being a diagonal matrix such that $(\boldsymbol{I}^+)_{ii} = 1$ if

$(\boldsymbol{L}^+)_{ii} \neq 0$, $(\boldsymbol{I}^+)_{ii} = 0$ otherwise. Thus, the perturbation matrix can be obtained by solving the following Lyapunov equation:

$$\boldsymbol{L}^T \boldsymbol{P} + \boldsymbol{P} \boldsymbol{L} = -(\boldsymbol{L}^{+T} + \boldsymbol{L}^+) - \varepsilon \boldsymbol{I}^+. \tag{4.4}$$

To this end, $\boldsymbol{L}^+$ and $\varepsilon$ must be specified first. As previously mentioned, $\boldsymbol{L}^+$ must contain the part of the linear operator $\boldsymbol{L}$ responsible for the transient growth of energy. For the DM model, under the assumptions made on the subspace $\mathscr{D}^-$ and, in particular, its invariance with respect to $\boldsymbol{L}$, the only possible choice is $\boldsymbol{L}^+ = \boldsymbol{L}$. Thus, the perturbation matrix is easily computed from (4.4) and is such that

$$\boldsymbol{M} = \boldsymbol{P} + \boldsymbol{I} = \begin{pmatrix} -\dfrac{\varepsilon}{2s_1} & \dfrac{\varepsilon}{2s_1(s_1+s_2)} \\ \dfrac{\varepsilon}{2s_1(s_1+s_2)} & -\dfrac{\varepsilon\left(1+s_1^2+s_1 s_2\right)}{2s_1 s_2(s_1+s_2)} \end{pmatrix}, \tag{4.5}$$

$\boldsymbol{M}$ being the matrix defining the new metric. It appears that $\boldsymbol{M}$ is linear in $\varepsilon$. In general, when $\boldsymbol{L}^+$ coincides with the whole linear operator, the perturbed metric turns out to be linear in $\varepsilon$. Thus, it can easily be seen that the results of the proposed stability criterion are independent of $\varepsilon$. For simplicity, let us thus consider here $\varepsilon = 1$. Equation (4.5) thus defines the generalized energy for all values of the DM model parameters, $s_1$ and $s_2$. We consider, first, a particular DM model (Cossu 2004), obtained for $s_1 = -(4Re)^{-1}$ and $s_2 = -(Re)^{-1}$, and we proceed with the solution of (2.3) for different values of $Re$. Recall that the solution of (2.3) gives the threshold amplitude $\alpha$ in terms of the new metric, and that the generalized energy of any disturbance whose initial amplitude is lower than $\alpha$ monotonically vanishes in time. In order to compare with the results in the literature, usually given in terms of critical energy or amplitude in the $L^2$ norm, these values must be computed from the critical amplitudes in the new norm. This can be done, since the following bound exists: $\mathscr{E} \geqslant \lambda_m E$, $\lambda_m$ being the maximum eigenvalue of $\boldsymbol{M}$. Thus, by taking

$$\sqrt{E_0} = \sqrt{1/\lambda_m}\,\alpha \tag{4.6}$$

an amplitude threshold can be obtained, which guarantees that all the disturbances with initial kinetic energy lower than $E_0$ are characterized by a monotonically decreasing generalized energy and are thus stable. This choice is conservative, but it introduces an additional source of error, as will be discussed in the following.

Figure 1($a$) compares the threshold amplitude (in the $L^2$ norm) obtained by the proposed criterion, for the DM model considered and for different Reynolds numbers, to the results of the computations in Cossu (2004), taken as a reference solution. Good agreement is found for $Re \geqslant 2$ and, in particular, the scaling of the critical amplitude for large enough Reynolds numbers, i.e. $Re^{-3}$, is predicted well. For $1 < Re < 2$ the proposed criterion underestimates the amplitude threshold in the $L^2$ norm. Recall that this DM model is globally stable for $Re \leqslant 1$. To better understand the behaviour of the proposed criterion, figures 1($b$) and 1($c$) show part of the line delimiting the attraction basin of the basic laminar flow (thick dashed line) together with the stability region given by the proposed criterion in the new metric (largest ellipse) at $Re = 1.1$ and $Re = 2$, respectively. The inscribed circle for this ellipse is the stability region obtained from the proposed criterion in the $L^2$ metric through (4.6), while the smallest ellipse is the stability region given by the energy criterion, when the 'exotic' energy proposed in Dauchot & Manneville (1997) is used. Note first that, due to the introduced perturbation of the $L^2$ metric, a constant level of generalized energy is identified in the phase space by an ellipse with axes rotated with respect to the axes

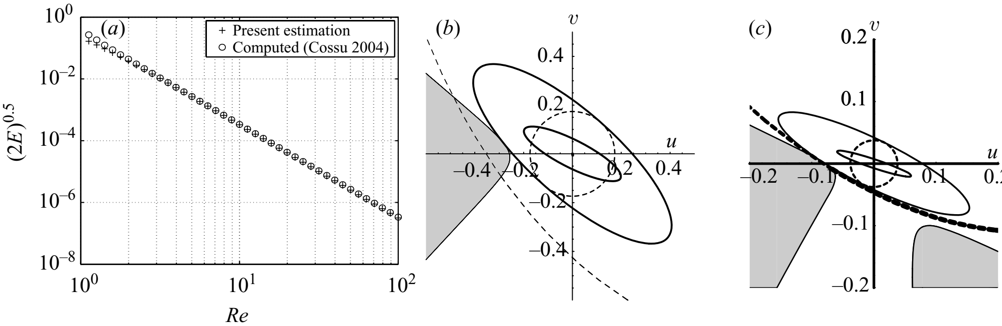


FIGURE 1. (*a*) Threshold amplitude (in the $L^2$ norm) of the initial critical disturbance given by the proposed energy criterion for the DM model considered, as a function of the Reynolds number. (*b*) Behaviour of the proposed criterion in phase space at $Re = 1.1$: part of the line delimiting the basin of attraction of the basic laminar flow (thick dashed line), stability region given by the proposed criterion in the new metric (largest ellipse) and in the $L^2$ metric (dashed circle), stability region given by an energy criterion using the 'exotic' energy of Dauchot & Manneville (1997) (smallest ellipse). In grey area $\mathrm{d}\mathscr{E}/\mathrm{d}t \geqslant 0$. (*c*) Same as (*b*) but for $Re = 2$.

$u = 0$ and $v = 0$. As stated in § 2, since the metric depends on the Reynolds number, the direction of the axes and the aspect ratio of the ellipse change with the Reynolds number (compare figures 1*b* and 1*c*). The critical disturbance is identified by the point at which the stability region given by the proposed criterion is tangent to the line delimiting the grey area, on which $\mathrm{d}\mathscr{E}/\mathrm{d}t = 0$, the grey area being the region of the phase space in which $\mathrm{d}\mathscr{E}/\mathrm{d}t \geqslant 0$. At $Re = 2$, this point belongs to the boundary of the actual attraction basin of the basic laminar flow, and thus an accurate prediction of the critical threshold amplitude is obtained in terms of the generalized energy. It can be seen that the estimate of the stability region in terms of $L^2$ energy, given by the circle, is also very close to the boundary of the stability region and, thus, the prediction of the threshold amplitude in the $L^2$ norm is also accurate (see figure 1*a*). At $Re = 1.1$, the stability regions given by the proposed criterion both in the new and in the $L^2$ metrics do not reach the limit of the basin of attraction; thus, the proposed criterion underestimates the critical amplitude threshold. However, note that the error in the estimation obtained in the new metric, i.e. the minimum distance between the largest ellipse and the dashed line in figure 1(*b*) is significantly lower than the error in the $L^2$ norm, i.e. the minimum distance between the circle and the dashed line. This is because, as previously mentioned, the proposed criterion naturally gives the critical energy threshold in the new metric; in order to express this in terms of classical energy, an inequality is used and a further error may be introduced. We repeated the procedure with different values of $s_1$ and $s_2$, always proportional to $-Re^{-1}$, and in all cases found that the critical amplitude scales as $Re^{-3}$, as indicated in Dauchot & Manneville (1997).

For both the W95 and W97 models, the variables involved are $u$, $v$, $w$, $m$. The linear operator in (4.1), for both W95 and W97, is given by

$$\boldsymbol{L} = \begin{pmatrix} -k_u^2/Re & \sigma_u & 0 & 0 \\ 0 & -k_v^2/Re & 0 & 0 \\ 0 & 0 & -\left(k_w^2/Re + \sigma_m\right) & 0 \\ 0 & 0 & 0 & -k_m^2/Re \end{pmatrix}.$$

Waleffe (1995, 1997) give the relevant expressions for the nonlinear term. The W95 model corresponds to $k_u = k_v = k_m = \sqrt{10}$, $k_w = \sqrt{15}$, $\sigma_u = \sigma_v = 1$, $\sigma_w = 1/2$ and $\sigma_m = 0$, and the W97 model to $k_u = \sqrt{5.2}$, $k_v = \sqrt{7.67}$, $k_m = \sqrt{2.47}$, $k_w = \sqrt{7.13}$, $\sigma_u = 1.29$, $\sigma_v = 0.22$, $\sigma_w = 0.68$ and $\sigma_m = 0.31$. Their dynamics are studied in Waleffe (1995, 1997). We just recall that the global stability Reynolds number is $Re_g = 98$ for W95 and $Re_g = 104.84$ for W97, and that, applying the classical energy theory, gives a critical energy Reynolds number, $Re_E$, equal to 20 for W95 and to 4.89 for W97.

For these models, the choice of $\boldsymbol{L}^+$ is not unique. We start by considering three possible choices satisfying the requirements on $\mathscr{D}^-$ (see § 3), and then we discuss how the results depend on $\varepsilon$. The first is $\boldsymbol{L}^+ = \boldsymbol{L}$ (case A). As a second case (B), $\boldsymbol{L}^+$ is obtained from $\boldsymbol{L}$ by putting $L_{33} = 0$, while in the last case (C) both $L_{33}$ and $L_{44}$ are set equal to zero. An additional choice would be possible, namely to obtain $\boldsymbol{L}^+$ from $\boldsymbol{L}$ by putting $L_{44} = 0$; however, this behaves very similarly to case B. By solving (4.4) with $\boldsymbol{L}^+$ thus defined as in cases A, B and C, the perturbation matrices defining the new metrics are computed. For case A we obtain

$$\boldsymbol{M}_A = \begin{pmatrix} \dfrac{Re\varepsilon}{2k_u^2} & \dfrac{Re^2\sigma_u\varepsilon}{2k_u^2(k_u^2+k_v^2)} & 0 & 0 \\ \dfrac{Re^2\sigma_u\varepsilon}{2k_u^2(k_u^2+k_v^2)} & \dfrac{Re(k_u^4+k_u^2k_v^2+Re^2\sigma_u^2)\varepsilon}{2k_u^2k_v^2(k_u^2+k_v^2)} & 0 & 0 \\ 0 & 0 & \dfrac{Re\varepsilon}{2k_w^2+2Re\sigma_m} & 0 \\ 0 & 0 & 0 & \dfrac{Re\varepsilon}{2k_m^2} \end{pmatrix}. \tag{4.7}$$

$\boldsymbol{M}_B$, defining the new metric for case B, is equal to $\boldsymbol{M}_A$ except that $(\boldsymbol{M}_B)_{33} = 1$, and $\boldsymbol{M}_C$ is equal to $\boldsymbol{M}_A$ except that $(\boldsymbol{M}_C)_{33} = (\boldsymbol{M}_C)_{44} = 1$. These matrices can be specialized either for W95 or W97 by inserting the relevant values of the model parameters. Going from case A to case C, the part of the time variation of the classical energy to be counteracted by the perturbation term decreases and, as a consequence, the perturbation introduced in the new metric, with respect to the $L^2$ one, is also expected to become smaller.

In all cases the parameter $\varepsilon$ is initially chosen to obtain $\det(\boldsymbol{I} + \boldsymbol{P}) = 1$, so that the metric defining the new energy has the same volume measure as the $L^2$ one.

In figure 2(*a*) the threshold amplitudes of the critical initial disturbance, obtained for W95 through the energy criterion proposed, are shown as a function of the Reynolds number; the threshold amplitudes are given in the $L^2$ norm. The scaling (for large enough Reynolds numbers) of the critical amplitude (in $L^2$) for the W95 model has been estimated as $Re^k$, with $k = -2$ (Baggett & Trefethen 1997; Cossu 2005). The amplitude threshold values computed by Cossu (2005), which indeed scale as $Re^{-2}$, are shown in figure 2(*a*) for comparison. It can be seen that the less 'aggressive' the metric perturbation the more accurate is the scaling of the critical amplitude with the Reynolds number. In particular, $k = -3, -2.4, -2$ is obtained for cases A, B and C respectively, and thus the correct scaling is predicted in case C. However, comparing the values obtained through the proposed criterion with those computed in Cossu (2005), shows that they are significantly underestimated, also for case C. This is also confirmed by numerical simulations, for a wide range of Reynolds numbers, carried

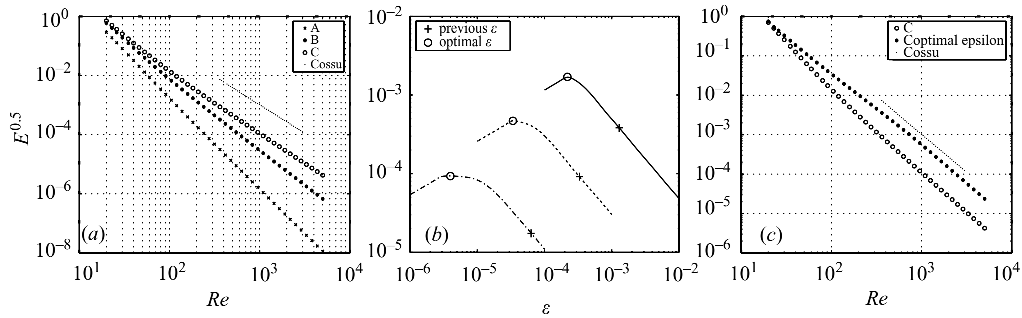


FIGURE 2. W95: ($a$) Threshold amplitudes (in the $L^2$ norm) given by the proposed criterion for cases A, B, C as a function of $Re$. ($b$) Kinetic energy threshold as a function of the parameter $\varepsilon$ for three values of $Re$: —, $Re=549$; - - -, $Re=1097$; - · - · - $Re=2507$. ($c$) Same as ($a$) but for case C with the previous and the optimal values of $\varepsilon$.

out using the critical disturbance shape as initial condition in order to measure the amplification factor needed to trigger instability. From these simulations for case C, the predicted critical disturbance appears to need an amplification factor ranging from 5 to 8 to become unstable.

Recall that those results were obtained with the parameter $\varepsilon$ fixed by requiring the volume measure of the new metric to be the same as that of the classic one; however, by applying the proposed energy criterion for different values of $\varepsilon$ we find out that the predicted critical energy threshold depends on $\varepsilon$ and that, for all $Re$, it is characterized by a maximum, as shown in figure 2($b$). The previously used values of $\varepsilon$ are also indicated by the crosses. Thus, the conditional stability curve has been computed again for case C using the value of the parameter $\varepsilon$ at which the critical $L^2$ energy threshold is a maximum (corresponding to the open circles in figure 2($b$) and hereafter called *optimal* $\varepsilon$). The energy threshold obtained has the same correct scaling with the Reynolds number, but gives a better quantitative areement with the results of the numerical simulations in Cossu (2005) (see figure 2$c$). This is also confirmed *a posteriori* by numerical simulation; the predicted critical disturbance is indeed found to be unstable, when multiplied by a factor that is very close to one ($1.06 \sim 1.30$).

The calculations carried out for the W97 model confirm that, as observed for the W95 case, the results that are closest to the reference simulations in Cossu (2005) are obtained for the least perturbative modification of the $L^2$ metric, corresponding to case C. However, for this model, even when using the *optimal* $\varepsilon$, the threshold amplitude in the $L^2$ norm is significantly underestimated: following our criterion it is well fitted by $3Re^{-1.3}$, while the simulations in Cossu (2004, 2005) give $15Re^{-1.07}$. This is because, although the energy transient growth caused by the non-normality of the linear operator has been eliminated, the region of the phase space in which the generalized energy decays monotonically is, in this case, well inside the actual attraction basin of the basic laminar solution. Note that, following our criterion, this is the region in which the contribution of the nonlinear term to the time derivative of $\mathscr{E}$ is smaller than that of the linear term.

## 5. Concluding remarks

The capabilities of an energy criterion for conditional stability, based on the definition of generalized energies, have been investigated. The procedure consists of two main steps. First, a generalized energy is defined through a perturbation of the classical $L^2$ energy, to by-pass the problems related to the non-normality of the linearized operator. Indeed, the pertubation is such that the time variation of the generalized energy due to the linear term of the disturbance evolution equation is negative definite. Second, by imposing the monotonic decay of the generalized energy, a criterion of conditional stability is obtained for each Reynolds number up to $Re_L$.

The procedure has been applied to three different low-dimensional systems. For the two-dimensional DM model the definition of the generalized energy is unique and the predictions of the proposed criterion are independent of the free parameter $\varepsilon$. The proposed criterion gives an estimate of the critical disturbance amplitude which is in excellent agreement with the reference simulations in Cossu (2004), except for very low Reynolds numbers ($Re < 2$). For the four-dimensional W95 and W97 models there are different possible choices in the construction of the generalized energy. For W95, the correct scaling of the critical disturbance amplitude with the Reynolds number has been obtained by constructing the generalized energy with the least perturbative modification of the $L^2$ metric. A satisfactory (although slightly underestimated) quantitative evaluation of the critical amplitude has also been obtained by adopting a particular choice of $\varepsilon$. The calculations for the W97 model confirmed that the least perturbative modification of the $L^2$ metric is the best choice, as expected since it corresponds to the case in which the contribution of the nonlinear term to the generalized energy growth is as small as possible. However, for this model, even with the *optimal* choice of $\varepsilon$, the threshold amplitude is significantly underestimated by the proposed criterion. This indicates that, although in our criterion the problem of the the non-normality of the linear operator has been overcome, the region in which the generalized energy monotonically decays may be well inside the actual stability region. This depends on the relative weight of the nonlinear and linear terms in the time evolution of the generalized energy, which suggests a possible modification of the construction of the generalized energy to impose a further constraint on the relative weights of the different contributions to the time derivative of $\mathscr{E}$.

Nonetheless, the present criterion gives for each $Re$ a region of the phase space in which the system is certainly stable, although this region may be smaller than the actual one. This information can be useful, for instance, when searching for the critical amplitude by numerical simulation, as in Cossu (2004, 2005); all the disturbances inside the stability region given by the present criterion can be excluded from the possible initial conditions, and the simulations can be stopped as soon as the disturbance enters this region.

Finally, the procedure is general and can be applied to any finite-dimensional dynamic system, and, in particular, to those deriving from the Galerkin projection of the Navier–Stokes equations, as in most numerical discretization methods. Following the present results, the generalized energy can be computed by choosing as $\mathbf{L}^+$ the smallest block of the linear operator matrix which causes the classical energy growth, and by numerically solving the relevant Lyapunov equation.

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## Appendix. Additional properties of the generalized energy

To any symmetric bilinear form $\mathscr{S}$ we can associate another bilinear form $\delta\mathscr{S}$, defined by

$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle_{\delta\mathscr{S}} = -(\langle L\boldsymbol{u}, \boldsymbol{v} \rangle_{\mathscr{S}} + \langle \boldsymbol{u}, L\boldsymbol{v} \rangle_{\mathscr{S}}). \tag{A 1}$$

This map is formally invertible as

$$\int_{t=0}^{\infty} \langle \mathrm{e}^{tL}\boldsymbol{u},\, \mathrm{e}^{tL}\boldsymbol{v}\rangle_{\delta\mathscr{S}}\,\mathrm{d}t = -\int_{t=0}^{\infty} \langle L\,\mathrm{e}^{tL}\boldsymbol{u},\, \mathrm{e}^{tL}\boldsymbol{v}\rangle_{\mathscr{S}} + \langle \mathrm{e}^{tL}\boldsymbol{u},\, L\,\mathrm{e}^{tL}\boldsymbol{v}\rangle_{\mathscr{S}}\mathrm{d}t$$
$$= -\int_{t=0}^{\infty} \frac{\mathrm{d}}{\mathrm{d}t}\left(\langle \mathrm{e}^{tL}\boldsymbol{u},\, \mathrm{e}^{tL}\boldsymbol{v}\rangle_{\mathscr{S}}\right)\mathrm{d}t = \langle \boldsymbol{u}, \boldsymbol{v}\rangle_{\mathscr{S}} \qquad \text{(A 2)}$$

where the last equality follows because $\boldsymbol{L}$ is assumed to have a stable spectrum so that $\lim_{t\to\infty} \mathrm{e}^{t\,L}\boldsymbol{u} = 0$, $\forall \boldsymbol{u} \in \mathscr{D}$. This means that in the proposed procedure, we always assume that $Re < Re_L$. Specializing (A 1) and (A 2) to the new energy, we have

$$\mathscr{L}(\boldsymbol{u}) = \langle \boldsymbol{u},\, \boldsymbol{L}\boldsymbol{u}\rangle_{\mathscr{E}} = -\tfrac{1}{2}\,\langle \boldsymbol{u}, \boldsymbol{u}\rangle_{\delta\mathscr{E}},\ \mathscr{E}(\boldsymbol{u}) = \tfrac{1}{2}\,\langle \boldsymbol{u}, \boldsymbol{u}\rangle_{\mathscr{E}} = \tfrac{1}{2}\int_{t=0}^{\infty} \left\langle e^{tL}\boldsymbol{u},\, \mathrm{e}^{tL}\boldsymbol{u}\right\rangle_{\delta\mathscr{E}}\,\mathrm{d}t. \qquad \text{(A 3)}$$

It follows that the problem of choosing the new energy functional can be reduced to the choice of a positive-definite bilinear form $\delta\mathscr{E}$. Thus, the required condition for $\mathscr{L}$ to be negative definite automatically implies that $\mathscr{E}$ is positive definite. From a practical viewpoint, this implies that, in constructing the new energy, we need only consider the satisfaction of the requirement on $\mathscr{L}$. Moreover, (A 3) has the following interesting interpretation: measuring the instantaneous value of the disturbance energy $\mathscr{E}$ is equivalent to evaluating the integral of its time evolution due to the linearized Navier–Stokes operator in the metric $\delta\mathscr{E}$. Thus, this allows information on the global time behaviour of the perturbation to be obtained by the evaluation of the $\mathscr{E}$ energy at the initial condition.

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